

Bilarge Neutrino Mixing and $\mu - \tau$ Permutation Symmetry for Two-loop Radiative Mechanism

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The presence of approximate electron number conservation and μ - τ permutation symmetry of S_2 is shown to naturally provide bilarge neutrino mixing. First, the bimaximal neutrino mixing together with $U_{e3}=0$ is guaranteed to appear owing to S_2 and ,then, the bilarge neutrino mixing together with $|U_{e3}| \ll 1$ arises as a result of tiny violation of S_2 . The observed mass hierarchy of $\Delta m_{\odot}^2 \ll \Delta m_{atm}^2$ is subject to another tiny violation of the electron number conservation. This scenario is realized in a specific model based on $SU(3)_L \times U(1)_N$ with two-loop radiative mechanism for neutrino masses. The radiative effects from heavy leptons contained in lepton triplets generate the bimaximal structure and those from charged leptons, which break S_2 , generate the bilarge structure together with $|U_{e3}| \ll 1$. To suppress dangerous flavor-changing neutral current interactions due to Higgs exchanges especially for quarks, this S_2 symmetry is extended to a discrete Z_8 symmetry, which also ensures the absence of one-loop radiative mechanism.

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I. INTRODUCTION

Recent observations of terrestrial neutrino oscillations at K2K [1] and KamLAND [2] are proved to be consistent with the observations of atmospheric and solar neutrino oscillations [3, 4]. These oscillations are characterized by the squared mass differences and mixing angles, which are respectively given by $\Delta m_{atm}^2 \sim 2 \times 10^{-3} \text{ eV}^2$ and $\sin^2 2\theta_{atm} \sim 1$ for atmospheric neutrinos and $\Delta m_{\odot}^2 \sim 7 \times 10^{-5} \text{ eV}^2$ and $\sin^2 2\theta_{\odot} \sim 0.8$ for solar neutrinos [5]. These mixings can be theoretically explained by the oscillations among three known neutrinos, $\nu_{e,\mu,\tau}$, if they are massive [6]. In fact, the experimental data indicate that the atmospheric and solar neutrino oscillations can, respectively, originate from the $\nu_{\mu}-\nu_{\tau}$ mixing and the $\nu_e-\nu_{\mu}$ mixing, which are both almost maximal. The mass hierarchy of $\Delta m_{atm}^2 \gg \Delta m_{\odot}^2$ as well as the large mixing angles suggest that the neutrino mass matrix has almost bimaximal structure [7, 8].

To accommodate massive neutrinos with tiny masses of $\mathcal{O}(10^{-1}) \text{ eV}$ ($\sim \sqrt{\Delta m_{atm}^2} \text{ eV}$), there are two theoretical ideas: one is the seesaw mechanism [9, 10] and the other is the radiative mechanism [11, 12]. While to understand the observed patterns of the neutrino mixings, let us consider, as the first approximation, the maximal neutrino mixings. To realize these maximal mixings, we often demand specific relations among neutrino masses, for example, $m_{\nu_e \nu_{\mu}} = m_{\nu_e \nu_{\tau}}$ and/or $m_{\nu_{\mu} \nu_{\mu}} = m_{\nu_{\tau} \nu_{\tau}}$, where $m_{\nu_i \nu_j}$ ($i, j=e, \mu, \tau$) stands for the mass for $\nu_i-\nu_j$. To naturally ensure the presence of such specific relations, one may invoke a certain symmetry such as a $U(1)$ symmetry based on $L_e - L_{\mu} - L_{\tau}$ ($\equiv L'$) [13] for the maximal solar neutrino mixing or a μ - τ permutation symmetry for the maximal atmospheric neutrino mixing [14, 15, 16, 17].

In this article, we consider the neutrino oscillation based on the radiative mechanism. It has been argued that the original version of the Zee's one-loop radiative mechanism [11] failed to explain the favorable LMA solution for solar neutrinos, namely, the significant deviation from the maximal solar neutrino mixing [18]. To get around the difficulty, one has to extend the original Zee's framework [14, 19] by 1) permitting the second Higgs scalar (introduced in the Zee model) to couple to the leptons, 2) employing a triplet Higgs scalar, 3) including a sterile neutrino or 4) introducing

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two-loop radiative effects with the absence of one-loop radiative ones. We are concentrating on examining two-loop radiative effects [12] since there are few theoretical models based on two-loop radiative mechanism that explain the observed properties of the current neutrino oscillations. The advantage of utilizing two-loop radiative effects lies in the fact that Majorana neutrino masses of $\mathcal{O}(10^{-1})$ eV can be naturally generated without strong fine-tuning of various couplings. Denoting the lepton-number violating coupling by f inherent to Majorana neutrino masses, we roughly estimate the one-loop effect to be $f m_\tau^2 \mu / 16\pi^2 M^2$ and the two-loop effect to be $f^2 m_\tau^2 \mu / (16\pi^2)^2 M^2$ for $\mu \sim v_{weak} (= (2\sqrt{2}G_F)^{-1/2} = 174 \text{ GeV})$ as the mass scale of the standard model and $M \sim 1 \text{ TeV}$ as the mass scale beyond the standard model. The estimated mass becomes $\mathcal{O}(10^{-1})$ eV for $f \sim 10^{-4}$ in the one-loop effect and $f \sim 10^{-1}$ in the two-loop effect. The very small magnitude of $f \sim 10^{-4}$ for the one-loop case entails the strong fine-tuning while the magnitude of $f \sim 0.1$ for the two-loop case can be naturally of order of e , where e stands for the electric charge. In the present analysis, all the new couplings f beyond the standard-model ones are set to be $\sim e$. The pattern of the neutrino mixings is also naturally explained by the use of the μ - τ permutation symmetry.

To respect the power of the permutation symmetry (without fine-tunings of couplings to meet an ‘‘artificial’’ permutation symmetry), since ordinary charged leptons badly break the permutation symmetry by their masses, new sources, whose interactions can preserve the permutation symmetry, are required. As the simplest extension of the standard model, we choose an $SU(3)_L \times U(1)_N$ model [20], which contains heavy leptons in lepton triplets as new sources. $SU(3)_L \times U(1)_N$ models are known to exhibit the intriguing aspect that

1. it predicts three families of quarks and leptons if the anomaly free condition on $SU(3)_L \times U(1)_N$ and the asymptotic free condition on $SU(3)_c$ are imposed;
2. the need of the three families of quarks and leptons group-theoretically arises because the anomalies from three families of leptons and one family of quarks that are $SU(3)_L$ -triplets are cancelled by those from two families of quarks that are $SU(3)_L$ -antitriplets.

Furthermore, these models naturally provide the radiative mechanism for neutrinos of the Majorana type because the standard Higgs doublet together with a Zee scalar specific to the one-loop radiative mechanism is regarded as the Higgs triplet [21, 22, 23]. Our $SU(3)_L \times U(1)_N$ model includes three families of lepton triplets to be denoted by $\psi_L^i = (\nu_L^i, \ell_L^i, E_L^i)^T$ ($i=e, \mu, \tau$), where $(\nu_L^i, \ell_L^i)^T$ is the standard doublet and E_L^i is negatively charged heavy leptons [24]. It can be argued that to activate a two-loop radiative mechanism requires a Higgs scalar of $\xi=(\xi^{++}, \xi^+, \bar{\xi}^+)^T$ [23]. It will be shown that Eq.(13) is generated by leading two-loop radiative effects due to the heavy-lepton exchanges while the charged-lepton exchanges add less-dominant two-loop effects that give a slight deviation of θ_{atm} from the maximal value of $\theta_{atm} = \pi/4$ and a significant contribution on θ_\odot to give $\sin^2 2\theta_\odot \sim 0.8$.

In the next section, we discuss the possible pattern of neutrino mass matrix, which is consistent with the observed neutrino oscillations. In Sec.III, we describe the detail of the $SU(3)_L \times U(1)_N$ model including Yukawa and Higgs interactions as well as the symmetry structure summarized in Sec.II. In Sec.III, how to generate the neutrino masses and mixings is discussed and the observed properties of neutrino oscillations are shown to be well explained. The final section is devoted to summary.

II. NEUTRINO MASS MATRIX

To see the appearance of underlying symmetries that approximately describe the observed neutrino oscillations, we show the parameterization of masses and mixing angles in terms of neutrino masses given by a matrix, M_ν ,

$$M_\nu = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \quad (1)$$

on the flavor basis. The digitalization of M_ν is performed by the mixing matrix of U_{MNS} [6] defined by

$$U_{MNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{23} & \sin \theta_{23} \\ 0 & -\sin \theta_{23} & \cos \theta_{23} \end{pmatrix} \cdot \begin{pmatrix} \cos \theta_{13} & 0 & \sin \theta_{13} \\ 0 & 1 & 0 \\ -\sin \theta_{13} & 0 & \cos \theta_{13} \end{pmatrix} \cdot \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\sin \theta_{12} & \cos \theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2)$$

$$= \begin{pmatrix} \cos \theta_{12} \cos \theta_{13} & \sin \theta_{12} \cos \theta_{13} & \sin \theta_{13} \\ -\cos \theta_{23} \sin \theta_{12} - \sin \theta_{23} \cos \theta_{12} \sin \theta_{13} & \cos \theta_{23} \cos \theta_{12} - \sin \theta_{23} \sin \theta_{12} \sin \theta_{13} & \sin \theta_{23} \cos \theta_{13} \\ \sin \theta_{23} \sin \theta_{12} - \cos \theta_{23} \cos \theta_{12} \sin \theta_{13} & -\sin \theta_{23} \cos \theta_{12} - \cos \theta_{23} \sin \theta_{12} \sin \theta_{13} & \cos \theta_{23} \cos \theta_{13} \end{pmatrix}. \quad (3)$$

The masses, $m_{1,2,3}$, and mixing angles, $\theta_{12,23,13}$, for $\theta_{\odot}=\theta_{12}$ and $\theta_{atm}=\theta_{23}$, are computed to be

$$\begin{aligned} m_1 &= c_{12}^2 \lambda_1 + s_{12}^2 \lambda_2 - 2c_{12}s_{12}A, & m_2 &= s_{12}^2 \lambda_1 + c_{12}^2 \lambda_2 + 2c_{12}s_{12}A, \\ m_3 &= s_{23}^2 d + c_{23}^2 f + 2c_{23}s_{23}e, \end{aligned} \quad (4)$$

$$\begin{aligned} \sin^2 2\theta_{12} &= \frac{8}{8+x^2}, & \sin^2 2\theta_{23} &= 1 - 4\Delta^2 \left(c_{23}^2 = \frac{1}{2} + \Delta, s_{23}^2 = \frac{1}{2} - \Delta \right), \\ \tan 2\theta_{13} &= 2 \frac{s_{23}b + c_{23}c}{m_3 - \lambda_1} \end{aligned} \quad (5)$$

with

$$\lambda_1 = a, \quad \lambda_2 = c_{23}^2 d + s_{23}^2 f - 2c_{23}s_{23}e, \quad A = \frac{c_{23}b - s_{23}c}{c_{13}}, \quad (6)$$

$$x = \sqrt{2} \frac{\lambda_2 - \lambda_1}{A}, \quad \Delta = \frac{\sigma(f-d) + \sqrt{2}(b-\sigma c)s_{13}}{4e - \sqrt{2}(b+\sigma c)s_{13}}, \quad (7)$$

for $\Delta^2 \ll 1$, where $c_a = \cos \theta_a$ and $s_a = \sin \theta_a$ for $a=12, 23, 13$, and $\sigma = \pm 1$ for $s_{23} = \pm |s_{23}|$.

Since $\sin^2 \theta_{13} \ll 1$ [25] is reported, let us first set $\sin \theta_{13}=0$. For the maximal atmospheric neutrino mixing characterized by $\Delta=0$, the relation of $d=f$ (corresponding to $m_{\nu_\mu \nu_\mu}=m_{\nu_\tau \nu_\tau}$) is required and can arise as a result of the μ - τ permutation symmetry while the maximal solar neutrino mixing characterized by $x=0$ arises from $a=d=e=f=0$, indicating the L' conservation. Our strategy used in this article to realize the bilarge neutrino mixing is to start with the maximal atmospheric neutrino mixing due to the μ - τ permutation symmetry. This symmetry allows us to set

$$a = 0, \quad c = \pm b, \quad f = d, \quad (8)$$

which are subject to

1. the L_e conservation for $a=0$;
2. a tiny breaking of the L_e conservation for $b \neq 0$;
3. the μ - τ permutation symmetry for $c=\pm b$ and $f=d$.

It should be noted that a further possible relation of $d = f = e$ (or $d = f = -e$) may arise from the Majorana mass term of $(\nu_\tau + \nu_\mu)^2$ (or $(\nu_\tau - \nu_\mu)^2$).

Restricting ourselves to the case of $d = f = e$, thus utilizing $(\nu_\tau + \nu_\mu)^2$, we find the following situations on the masses and mixing angles of neutrinos. In the case of $c_{23} > 0$ and $s_{23} > 0$, the mass matrix with $c = b$ yields

$$m_1 = m_2 = 0, \quad m_3 = 2d, \quad \sin^2 2\theta_{23} = 1, \quad \tan 2\theta_{13} = \sqrt{2} \frac{b}{d}, \quad (9)$$

where θ_{12} is undetermined, while the mass matrix with $c = -b$ yields

$$|m_1| = |m_2| = \sqrt{2}b, \quad m_3 = 2d, \quad \sin^2 2\theta_{23} = \sin^2 2\theta_{12} = 1, \quad \sin 2\theta_{13} = 0. \quad (10)$$

In the case of $c_{23} > 0$ and $s_{23} < 0$, the mass matrix with $c = b$ yields

$$m_3 = 0, \quad \sin^2 2\theta_{23} = 1, \quad \sin^2 2\theta_{12} = \frac{8}{8+x^2} \left(x = \frac{2d}{b} \right), \quad \sin 2\theta_{13} = 0, \quad (11)$$

where m_1 and m_2 are given by Eq.(4) with θ_{12} given here, while the mass matrix with $c = -b$ yields

$$m_1 = \sin^2 \theta_{12} d, \quad m_2 = \cos^2 \theta_{12} d, \quad m_3 = 0, \quad \sin^2 2\theta_{23} = 1, \quad \sin 2\theta_{13} = 1, \quad (12)$$

where θ_{12} is undetermined. The gross feature of the observed neutrino oscillations is compatible with the zero-th order predictions in the cases of $s_{23} > 0$ with $b = -c$ [14, 15] and of $s_{23} < 0$ with $c = b$ [16].

We choose the case of $s_{23} > 0$ with $c = -b$ that incorporates the bimaximal structure:

$$M_\nu = \begin{pmatrix} 0 & b & -b \\ b & d & d \\ -b & d & d \end{pmatrix}. \quad (13)$$

The hierarchy of $|b| \ll |d|$, which is the seed of $\Delta m_{\odot}^2 \ll \Delta m_{atm}^2$ realized after $|m_1| \neq |m_2|$, originates from the tiny violation of the electron number conservation. The observed property of $\sin^2 2\theta_{\odot} \sim 0.8$ is explained by tiny violation of S_2 . In the next section, we realize this starting neutrino mass texture in a $SU(3)_L \times U(1)_N$ gauge model, where mass texture of heavy leptons as the third member of a lepton triplet plays an essential rôle.

III. MODEL

The particles in our model are specified by quantum numbers placed as $(SU(3)_L, U(1)_N)$. The hypercharge Y and the electric charge Q are given by $Y = \lambda^8/\sqrt{3} + N$ and $Q = (\lambda^3 + Y)/2$ respectively, where $N/2$ is the $U(1)_N$ number and λ^a are the Gell-Mann matrices with $\text{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$ ($a, b = 1, 2, \dots, 8$). Leptons are assigned to be:

$$\psi_L^i = (\nu^i, \ell^i, E^i)_L^T : (\mathbf{3}, -2/3), \quad \ell_R^{e,\mu,\tau} : (\mathbf{1}, -1), \quad E_R^{e,-,+} : (\mathbf{1}, -1), \quad (14)$$

where E_R^j for $j = (e, -, +)$ are the mass eigenstates of the negatively charged heavy leptons and the superscripts, \pm , of $E_{\pm R}$ represent the chiral partners of the $\tau \pm \mu$ states of $E_{\pm L}$ to be defined by their Yukawa interactions while quarks are assigned to be:

$$Q_L^e = (d, -u, u')_L^T : (\mathbf{3}^*, 1/3), Q_L^\mu = (s, -c, c')_L^T : (\mathbf{3}^*, 1/3), Q_L^\tau = (t, b, b')_L^T : (\mathbf{3}, 0), \\ u_R, c_R, t_R : (\mathbf{1}, 2/3), d_R, s_R, b_R : (\mathbf{1}, -1/3), u'_R, c'_R : (\mathbf{1}, 2/3), b'_R : (\mathbf{1}, -1/3). \quad (15)$$

Another notations for Q^i , $Q^i = (d^i, u^i, u'^i)_L^T$ ($i=e, \mu$) and $Q^\tau = (u^\tau, d^\tau, d'^\tau)_L^T$, and for the corresponding right-handed quarks are also used. All gauge anomalies are cancelled by these quarks and leptons [24]. Higgs scalars are assigned to be:

$$\eta = (\eta^0, \eta^-, \bar{\eta}^-)^T : (\mathbf{3}, -2/3), \quad \rho = (\rho^+, \rho^0, \bar{\rho}^0)^T : (\mathbf{3}, 1/3), \quad \chi = (\chi^+, \bar{\chi}^0, \chi^0)^T : (\mathbf{3}, 1/3), \quad (16)$$

which develop the following vacuum expectation values (VEV's):

$$\langle 0|\eta|0\rangle = (v_\eta, 0, 0)^T, \quad \langle 0|\rho|0\rangle = (0, v_\rho, 0)^T, \quad \langle 0|\chi|0\rangle = (0, 0, v_\chi)^T, \quad (17)$$

where the orthogonal choice of these VEV's will be guaranteed by appropriate Higgs interactions introduced in Eq.(26).

Since the model respects the μ - τ permutation symmetry of S_2 , in order to generate the phenomenologically consistent charged-lepton masses that do not respect the μ - τ permutation symmetry, the model employs the following S_2 -antisymmetric Higgs scalars:

$$\rho' = (\rho'^+, \rho'^0, \bar{\rho}'^0)^T : (\mathbf{3}, 1/3), \quad \rho'' = (\rho''^+, \rho''^0, \bar{\rho}''^0)^T : (\mathbf{3}, 1/3), \quad (18)$$

as well as S_2 -symmetric Higgs scalar, χ' :

$$\chi' = (\chi'^+, \bar{\chi}'^0, \chi'^0)^T : (\mathbf{3}, 1/3) \quad (19)$$

with

$$\langle 0|\rho'|0\rangle = (0, v_{\rho'}, 0)^T, \quad \langle 0|\rho''|0\rangle = (0, v_{\rho''}, 0)^T, \quad \langle 0|\chi'|0\rangle = (0, 0, v_{\chi'})^T. \quad (20)$$

The two-loop radiative mechanism can be initiated by introducing three Higgs scalars denoted by ξ' , k^{++} and k'^{++} as S_2 -symmetric states and ξ as an S_2 -antisymmetric state:

$$\xi = (\xi^{++}, \xi^+, \bar{\xi}^+)^T : (\mathbf{3}, 4/3), \quad \xi' = (\xi'^{++}, \xi'^+, \bar{\xi}'^+)^T : (\mathbf{3}, 4/3), \quad k^{++}, k'^{++} : (\mathbf{1}, 2), \quad (21)$$

where ξ' has $L_e=1$ and others have $L_e=0$.

Since our model contains quarks with the same charge, whose mass terms can be generated by ρ and χ between Q_L^e and down-type quarks and by ρ^\dagger and χ^\dagger between $Q_L^{\mu,\tau}$ and up-type quarks, dangerous flavor-changing-neutral-currents (FCNC) interactions are generally induced at the phenomenologically unacceptable level [27]. To avoid these interactions, Yukawa interactions must be constrained such that a triplet quark flavor gains a mass from only one Higgs field [28]. The lepton sector also contains the similar FCNC problem because ℓ^i ($i=e, \mu, \tau$) and E^i ($i=e, \pm$) has the same charge. It is known that to impose such a constraint on FCNC is readily achieved by introducing a certain discrete symmetry. We combine the S_2 -permutation symmetry and the discrete symmetry for the FCNC-suppression into a Z_8 symmetry. Listed in Table I are the quantum numbers of S_2 , Z_8 , L and L_e for leptons and Higgs scalars in our discussions, where $\psi_{\pm L} = (\psi_L^\tau \pm \psi_L^\mu)/\sqrt{2}$, $\ell_{\pm R} = (\tau_R \pm \mu_R)/\sqrt{2}$ and S_2 is shown for comparison. In this table, we have omitted the quark sector of the model since Z_8 of quarks can be easily adjusted to respect the Z_8 conservation for the given Z_8 of Higgs scalars to reach quark interactions in Eq.(22).

The Yukawa interactions for leptons are now described by \mathcal{L}_Y :

$$\begin{aligned}
-\mathcal{L}_Y = & \overline{(\psi_{+L})^c} (f_{\xi'} \psi_L^e \xi' + f_{\xi} \psi_{-L} \xi) \\
& + \left[f_k^{+\ell} (\ell_{+R})^c \ell_{+R} + \frac{1}{2} f_k^{-E} (\overline{E_{-R}})^c E_{-R} \right] k^{++} + \left[f_{k'}^{-\ell} (\ell_{-R})^c \ell_{-R} + \frac{1}{2} f_{k'}^{+E} (\overline{E_{+R}})^c E_{+R} \right] k'^{++} \\
& + f_{\ell} \overline{\psi_L^e} \rho e_R + \overline{\psi_L^+} (f_{\ell}^+ \rho \ell_{+R} + f_{\ell}^- \rho' \ell_{-R}) + \overline{\psi_L^-} (g_{\ell}^+ \rho'' \ell_{+R} + g_{\ell}^- \rho \ell_{-R}) \\
& + f_E \overline{\psi_L^e} \chi E_R^e + f_E^+ \overline{\psi_{+L}} \chi E_{+R} + f_E^- \overline{\psi_{-L}} \chi' E_{-R} \\
& + \sum_{i=e,\mu} \overline{Q_L^i} (\eta^c D_R^i + \rho^c U_R^i + \chi^c U_R^e) + \overline{Q_L^e} (\eta U_R^{\tau} + \rho D_R^{\tau} + \chi D_R^{\tau'}) + (h.c.)
\end{aligned} \tag{22}$$

with

$$U_R^i = \sum_{j=e,\mu,\tau} f_{uj}^i u_R^j, \quad D_R^i = \sum_{j=e,\mu,\tau} f_{dj}^i d_R^j, \quad U_R^i = \sum_{j=e,\mu} f_{w'j}^i u_R^j, \quad D_R^i = f_{d'\tau}^i d_R^{\tau}, \tag{23}$$

where f 's and g 's denote the Yukawa couplings. The Higgs interactions are given by Hermitian terms composed of $\phi_{\alpha}^{\dagger} \phi_{\beta}$ ($\phi = \eta, \rho, \rho', \rho'', \chi, \chi', \xi, \xi', k^{++}$), which include the potential terms of $V_{\eta\rho\chi, \rho\chi}$

$$V_{\eta\rho\chi} = \lambda_{\eta\rho} |\eta \times \rho|^2 + \lambda_{\rho\chi} |\rho \times \chi|^2 + \lambda_{\chi\eta} |\chi \times \eta|^2, \tag{24}$$

$$V_{\rho\chi} = \lambda_{\rho\rho'} |\rho^{\dagger} \rho'|^2 + \lambda_{\rho\rho''} |\rho^{\dagger} \rho''|^2 + \lambda_{\chi\chi'} |\chi^{\dagger} \chi'|^2 \tag{25}$$

with the definition of $(a \times b)^{\alpha} \equiv \epsilon^{\alpha\beta\gamma} a_{\beta} b_{\gamma}$ and by non-Hermitian terms in

$$\begin{aligned}
V = & \frac{1}{2} \lambda_{\eta\chi\xi\chi'} [(\eta^{\dagger} \chi)(\xi^{\dagger} \chi') + (\eta^{\dagger} \chi')(\xi^{\dagger} \chi)] + \lambda_{\eta\rho\xi\rho} (\eta^{\dagger} \rho)(\xi^{\dagger} \rho) + \frac{1}{2} \lambda_{\eta\rho'\xi\rho''} [(\eta^{\dagger} \rho')(\xi^{\dagger} \rho'') + (\eta^{\dagger} \rho'')(\xi^{\dagger} \rho')] \\
& + \mu_{\xi\eta k} \xi^{\dagger} \eta k^{++} + \mu_{\xi'\eta k'} \xi'^{\dagger} \eta k'^{++} + (h.c.),
\end{aligned} \tag{26}$$

where μ 's and λ 's denote mass scales and coupling constants, respectively. These Yukawa and Higgs interactions are invariant under

- the Z_8 -transformation, which is spontaneously broken;
- the L_e -transformation, which is not spontaneously broken but explicitly broken by $\xi'^{\dagger} \eta k'^{++}$.

The S_2 -permutation symmetry is preserved in the Yukawa interactions but explicitly broken by the Higgs interactions. The b - and c -terms in Eq.(1) can be generated by interactions containing $\xi'^{\dagger} \eta k'^{++}$. All other interactions are forbidden. We note that

- the orthogonal choice of VEV's of η , ρ and χ as in Eq.(17) is supported by $V_{\eta\rho\chi}$ if all λ 's are negative because $V_{\eta\rho\chi}$ gets lowered if η , ρ and χ develop VEV's and one can choose VEV's such that $\langle 0|\eta_1|0\rangle \neq 0$, $\langle 0|\rho_2|0\rangle \neq 0$ and $\langle 0|\chi_3|0\rangle \neq 0$;
- the choice of VEV's of ρ' , ρ'' and χ' as in Eq.(20) is supported by $V_{\rho\chi}$ if all λ 's are negative;
- a $(\eta^{\dagger} \chi)(\xi^{\dagger} \rho)$ -term, which is absent in V , would cause one-loop radiative effects such as in Fig.1 and similarly for the terms with $\rho \rightarrow \rho', \rho''$ and $\chi \rightarrow \chi'$;
- the suppression of FCNC for quarks is ensured by the absence of χD_R^e and ρD_R^e for Q_L^e , of $\chi^* U_R^i$ and $\rho^* U_R^i$ for Q_L^i and of $\chi \ell_R^i$ and ρE_R^i for ψ_L^i .

Our model also respects

- the L -conservation if $\lambda_{\eta\chi\xi\chi'}$, $\lambda_{\eta\rho\xi\rho}$ and $\lambda_{\eta\rho'\xi\rho''}$ are absent;
- the L' -conservation if $\mu_{\xi'\eta k'}$ is also absent.

Both the L - and L' -conservations are not spontaneously broken.

Before discussing neutrino oscillations, we should mention another dangerous flavor-changing interactions of leptons due to the existence of ρ , ρ' and ρ'' because the charged leptons can simultaneously couple to these Higgs scalars. These interactions are found to be suppressed down to the phenomenologically acceptable level because the suppression factor is at most m_{τ}^2/v_{weak}^2 from the incoming and outgoing vertices and provides enough suppression. Since the

approximate L_e conservation is satisfied by our interactions, all L_e -changing flavor interactions such as $\mu \rightarrow e\gamma$ are more suppressed. The detailed analyses have been described in Ref.[14]. The existence of heavy leptons and extra gauge bosons as well as heavy exotic quarks [29] may also disturb well-established weak-interaction phenomenology. However, the additional contributions from these exotics will be well-suppressed because they are sufficiently heavy. Their masses controlled by the VEV's of χ and χ' are taken to be ~ 2 TeV for the later analyses.

The heavy lepton mass matrix is simply given by the diagonal masses computed to be $m_{E^e} = f_E v_\chi$, $m_{E^+} = f_E^+ v_\chi$ and $m_{E^-} = f_E^- v_{\chi'}$. On the other hand, the charged lepton mass matrix has the following non-diagonal form:

$$M_\ell = \begin{pmatrix} m_\ell^{ee} & 0 & 0 \\ 0 & m_\ell^{\mu\mu} & m_\ell^{\mu\tau} \\ 0 & m_\ell^{\tau\mu} & m_\ell^{\tau\tau} \end{pmatrix}, \quad (27)$$

where

$$\begin{aligned} m_\ell^{ee} &= f_e v_\rho (= m_e), \\ m_\ell^{\mu\mu} &= \frac{1}{2} [(f_\ell^+ + g_\ell^-) v_\rho - f_\ell^- v_{\rho'} - g_\ell^+ v_{\rho''}], \quad m_\ell^{\mu\tau} = \frac{1}{2} [(f_\ell^+ - g_\ell^-) v_\rho + f_\ell^- v_{\rho'} - g_\ell^+ v_{\rho''}], \\ m_\ell^{\tau\mu} &= \frac{1}{2} [(f_\ell^+ - g_\ell^-) v_\rho - f_\ell^- v_{\rho'} + g_\ell^+ v_{\rho''}], \quad m_\ell^{\tau\tau} = \frac{1}{2} [(f_\ell^+ + g_\ell^-) v_\rho + f_\ell^- v_{\rho'} + g_\ell^+ v_{\rho''}]. \end{aligned} \quad (28)$$

The diagonalized charged lepton matrix, M_ℓ^{diag} , are obtained by the use of two unitary matrices of U and V as shown in the Appendix. For the given mixing angles of α and β in Eq.(A1), the mass parameters in Eq.(28) can be determined by Eq.(A5) in terms of $m_{\mu,\tau}$.

IV. NEUTRINO MASSES AND OSCILLATIONS

The neutrino mass matrix is given by M_ν^H and M_ν^C :

$$M_\nu = M_\nu^H + M_\nu^C, \quad (29)$$

where M_ν^H (M_ν^C) arises from the two-loop radiative mechanism based on Fig.2 (Fig.3) for the heavy-lepton (charged-lepton) exchanges. The effective mass terms are expressed by

$$(\psi_{+L}\rho\chi')^2, \quad \psi_{eL}\rho\chi' \cdot \psi_{-L}\rho\chi', \quad (30)$$

from Fig.2 and by

$$(\psi_{-L}\rho + \psi_{+L}\rho'')\chi \cdot (\psi_{-L}\rho + \psi_{+L}\rho'')\chi' + (\chi \leftrightarrow \chi'), \quad \psi_{eL}\rho\chi \cdot (\psi_{-L}\rho + \psi_{+L}\rho'')\chi' + (\chi \leftrightarrow \chi'), \quad (31)$$

from Fig.3, where the product of the three particles reads $abc = \epsilon^{\alpha\beta\gamma} a_\alpha b_\beta c_\gamma$. After, χ , χ' , ρ and ρ'' acquire VEV's, where some of these contribute to masses of the heavy leptons and of the charged leptons, the heavy-lepton contributions supply main terms of neutrino masses, which are either S_2 -symmetric or S_2 -antisymmetric, while the charged-lepton contributions supply minor terms of neutrino masses, which are the admixtures of the S_2 -symmetric and S_2 -antisymmetric contributions. These neutrino masses are expressed as:

$$M_\nu^H = m_\nu^H \begin{pmatrix} 0 & -r_\nu^H & r_\nu^H \\ -r_\nu^H & 1 & 1 \\ r_\nu^H & 1 & 1 \end{pmatrix}, \quad M_\nu^C = \begin{pmatrix} 0 & \delta m_\nu^{e\mu} & \delta m_\nu^{e\tau} \\ \delta m_\nu^{e\mu} & \delta m_\nu^{\mu\mu} & \delta m_\nu^{\mu\tau} \\ \delta m_\nu^{e\tau} & \delta m_\nu^{\mu\tau} & \delta m_\nu^{\tau\tau} \end{pmatrix}, \quad (32)$$

where M_ν^H and r_ν^H are calculated to be:

$$\begin{aligned} m_\nu^H &= -\frac{1}{2} f_\xi^2 (\lambda_{\eta\rho\xi\rho} v_\rho^2 + \lambda_{\eta\rho'\xi\rho''} v_{\rho'} v_{\rho''}) \mu_{\xi\eta k} m_{E^-}^2 - I_H (m_\xi^2, m_{E^-}^2, m_{k^{++}}^2), \\ r_\nu^H &= \sqrt{2} \frac{f_{\xi'} \mu_{\xi'\eta k'}}{f_\xi \mu_{\xi\eta k}} \frac{f_{k'}^{+E} m_{E^+}^2}{f_k^{-E} m_{E^-}^2} \frac{I_H (m_{\xi'}^2, m_{E^+}^2, m_{k'^{++}}^2)}{I_H (m_\xi^2, m_{E^-}^2, m_{k^{++}}^2)} \end{aligned} \quad (33)$$

with

$$I_H(m^2, m_E^2, m_k^2) = \frac{J(m_{E-}^2, m^2, m_E^2, m_{\eta-}^2; m_k^2) - J(m_E^2, m^2, m_{E-}^2, m_{\xi+}^2; m_k^2)}{m_{\eta-}^2 - m_{\xi+}^2}, \quad (34)$$

$$J(m_1^2, m_2^2, m_3^2, m_4^2; m_k^2) = \frac{G(m_1^2, m_2^2) G(m_3^2, m_4^2)}{m_k^2}, \quad (35)$$

$$G(x, y) = \frac{1}{16\pi^2} \frac{x \ln(x/m_k^2) - y \ln(y/m_k^2)}{x - y}, \quad (36)$$

under the assumption that the masses of k^{++} and k'^{++} are heavy enough to neglect other masses in the integration. The masses used here are denoted by m_ϕ^2 for the corresponding particle of ϕ . The explicit form of J is given by

$$J(a, b, c, d; e) = \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{1}{(k^2 - a)(k^2 - b)(q^2 - c)(q^2 - d)((k - q)^2 - e)}, \quad (37)$$

which can be evaluated via the following formula

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 - a)(q^2 - b)(q^2 - c)} = -\frac{i}{16\pi^2} \left[\frac{a \ln a}{(a - b)(a - c)} + \frac{b \ln b}{(b - a)(b - c)} + \frac{c \ln c}{(c - a)(c - b)} \right]. \quad (38)$$

To evaluate M_ν^C , it is convenient to express $-\mathcal{L}_Y$ of Eq.(22) in terms of the neutrinos in weak interactions and the diagonalized charged leptons, $\psi_\ell = (e_L, \mu_L, \tau_L)^T$, $\nu = (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})^T$ and $\ell = (e_R, \mu_R, \tau_R)^T$, whose relevant terms are:

$$\overline{\nu^c} U^T \mathbf{f}_\xi U \psi_\ell \bar{\xi}^+ + \overline{\nu^c} U^T \mathbf{f}_{\xi'} U \psi_\ell \bar{\xi}^+ + \overline{\ell^c} V^T \mathbf{f}_k V \ell k^{++} + \overline{\ell^c} V^T \mathbf{f}_{k'} V \ell k'^{++} + \overline{\psi}_\ell M_\ell^{diag} \ell + (\text{h.c.}), \quad (39)$$

where

$$\mathbf{f}_\xi = \frac{f_\xi}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{f}_{\xi'} = \frac{f_{\xi'}}{\sqrt{2}} \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{f}_k = \frac{f_k^{+\ell}}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{f}_{k'} = \frac{f_{k'}^{+\ell}}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \quad (40)$$

It is not difficult to reach an effective lagrangian corresponding to Fig.3:

$$\begin{aligned} \mathcal{L}_{eff} = & \lambda_{\eta\chi\xi\chi'} v_\chi v_{\chi'} \sum_{(\varphi, K)=(\xi, k), (\xi', k')} \mu_{\varphi\eta K} \sum_{m, m', n, n'=e, \mu, \tau} \overline{\nu_m^c} (U^T \mathbf{f}_\varphi^T U)_{mm'} (M_{diag}^\ell)_{m'm'} \\ & \times (V^T \mathbf{f}_K V)_{m'n'} I_\ell(m_{\varphi+}^2, m_{K++}^2)_{m'n'} (M_{diag}^\ell)_{n'n'} (U^T \mathbf{f}_\xi U)_{n'n} \nu_n, \end{aligned} \quad (41)$$

where

$$I_\ell(m_{\varphi+}^2, m_K^2)_{ab} = \frac{J(m_a^2, m_{\varphi+}^2, m_b^2, m_{\eta-}^2; m_K^2) - J(m_a^2, m_{\varphi+}^2, m_b^2, m_{\xi+}^2; m_K^2)}{m_{\eta-}^2 - m_{\xi+}^2}. \quad (42)$$

Eq.(41) can be further calculated to be:

$$\begin{aligned} \mathcal{L}_{eff} = & \frac{1}{16} f_\xi^2 f_k^{+\ell} \lambda_{\eta\chi\xi\chi'} v_\chi v_{\chi'} \mu_{\xi\eta k} I_\ell(m_{\xi+}^2, m_{k++}^2) \overline{\nu^c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_\tau^2 (c_\beta + s_\beta)^2 & -m_\mu m_\tau (c_\beta^2 - s_\beta^2) \\ 0 & -m_\mu m_\tau (c_\beta^2 - s_\beta^2) & m_\mu^2 (c_\beta - s_\beta)^2 \end{pmatrix} \nu \\ & - \frac{1}{16\sqrt{2}} f_\xi f_{\xi'} f_{k'}^{-\ell} \lambda_{\eta\chi\xi\chi'} v_\chi v_{\chi'} \mu_{\xi'\eta k'} I_\ell(m_{\xi'+}^2, m_{k'++}^2) \overline{\nu^c} \begin{pmatrix} 0 & X & Y \\ X & 0 & 0 \\ Y & 0 & 0 \end{pmatrix} \nu, \end{aligned} \quad (43)$$

where the integral of I_ℓ is defined as $I_\ell(m_1^2, m_2^2)_{ab} = \delta_{ab} I_\ell(m_1^2, m_2^2)$ with $m_{e, \mu, \tau}^2$ safely neglected in the denominators in the integral, and

$$\begin{aligned} X &= m_\tau^2 (c_\alpha + s_\alpha) (c_\beta - s_\beta)^2 - m_\mu m_\tau (c_\alpha - s_\alpha) (c_\beta^2 - s_\beta^2), \\ Y &= m_\mu m_\tau (c_\alpha + s_\alpha) (c_\beta^2 - s_\beta^2) - m_\mu^2 (c_\alpha - s_\alpha) (c_\beta + s_\beta)^2. \end{aligned} \quad (44)$$

Since

$$\mathcal{L}_{eff} = -\frac{1}{2}\bar{\nu}^c M_\nu^C \nu, \quad (45)$$

the each entry of δm_ν 's in Eq.(32) can be easily read off from Eq.(43).

To see the order of magnitudes of neutrino masses, we use the simplest choice of masses

$$m_{e,\mu,\tau}^2 \ll m_{\eta^-}^2 \ll m_{E^-,E^+}^2 \ll m_{\xi^+,\xi'^+,\bar{\xi}^+,\bar{\xi}'^+}^2 (= m^2) \ll m_{k^{++},k'^{++}}^2. \quad (46)$$

This choice reduces the integrals of I_H and I_ℓ to

$$I_H(m^2, m_E^2, m_k^2) = -\left(\frac{1}{16\pi}\right)^2 \frac{1}{m_k^2 m^2} \ln \frac{m_k^2}{m^2} \ln \frac{m^2}{m_E^2}, \quad I_\ell(m^2, m_k^2) = -\left(\frac{1}{16\pi}\right)^2 \frac{1}{m_k^2 m^2} \ln \frac{m_k^2}{m^2} \ln \frac{m^2}{m_{\eta^-}^2}. \quad (47)$$

We also choose the following magnitudes of relevant parameters for neutrino masses:

1. $v_\rho = v_{\rho'} = v_{\rho''} = v_{weak}/20$ (~ 10 GeV) and $v_\eta = v_{weak}$, where v_ρ that controls the b -quark mass can be safely set to be ~ 10 GeV and v_η that supplies the t -quark mass should be $\sim v_{weak}$, since v_η and $v_{\rho,\rho',\rho''}$ are related to weak boson masses proportional to $v_{weak}^2 = \sum_{all} v_{Higgs}^2$.
2. $v_\chi = v'_\chi = 10v_{weak}$ (~ 2 TeV) since χ and χ' are the key fields for the symmetry breaking of $SU(3)_L \times U(1)_N \rightarrow SU(2)_L \times U(1)_Y$, leading to $v_{\chi,\chi'} \gg v_{weak}$.
3. $m_{\eta^-} = v_{weak}$, $m_{E^\pm} = ev_\chi$ (~ 0.5 TeV), $m_{\xi^-} = m_{\bar{\xi}^-} = m_{\xi'^-} = m_{\bar{\xi}'^-} = v_\chi$ (~ 2 TeV) and $m_{k^{++},k'^{++}} = v_\chi/e$ (~ 6 TeV) to satisfy Eq.(46).
4. $\mu_{\xi\eta k} = ev_\chi$ with $\mu_{\xi'\eta k'}/\mu_{\xi\eta k} \sim 0.1$, which can be regarded as a natural relation [30] because the limit of $\mu_{\xi'\eta k'} \rightarrow 0$ recovers the L_e -conservation.
5. $f_{\xi,\xi'} = f_k^{-E} = f_{k'}^{+E} = -f_k^{+\ell} = -f_{k'}^{-\ell} = e$ and $\lambda_{\eta\rho\xi\rho,\eta\rho'\xi\rho'',\eta\chi\xi\chi'} \sim e$ so that higher loop effects are safely neglected.

This mass-setting provides sufficient suppression of exotic contributions in low-energy phenomenology.

To see the magnitude of parameters for neutrino oscillations, we omit the terms proportional to m_μ ($\ll m_\tau$) in Eq.(43) and obtain that

$$m_\nu^H = 3.3 \times 10^{-2} \frac{\lambda}{e} \text{ eV}, \quad r_\nu^H = \sqrt{2} \frac{\mu_{\xi'\eta k'}}{\mu_{\xi\eta k}}, \quad (48)$$

$$\delta m_\nu^{\mu\mu} = -6.6 \times 10^{-3} \frac{\lambda_\ell}{e} (c_\beta + s_\beta)^2 \text{ eV}, \quad \delta m_\nu^{e\mu} = \frac{6.6 \times 10^{-3}}{\sqrt{2}} \frac{\lambda_\ell}{e} \frac{\mu_{\xi'\eta k'}}{\mu_{\xi\eta k}} (c_\beta - s_\beta)^2 (c_\alpha + s_\alpha) \text{ eV}, \quad (49)$$

where $\lambda = \lambda_{\eta\rho\xi\rho} = \lambda_{\eta\rho'\xi\rho''}$ and $\lambda_\ell = \lambda_{\eta\chi\xi\chi'}$. By choosing the remaining parameters to be:

$$\frac{\lambda}{e} = 0.8, \quad \frac{\lambda_\ell}{\lambda} = 2.0, \quad \frac{\mu_{\xi'\eta k'}}{\mu_{\xi\eta k}} = 0.12, \quad (50)$$

so as to recover the observed data, we find using the estimates of Eqs.(48) and (49) that, for the simplest case of $s_{\alpha,\beta} = 0$,

$$\begin{aligned} \Delta m_{atm}^2 &= 2.2 \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{atm} = 0.96, \\ \Delta m_\odot^2 &= 7.3 \times 10^{-5} \text{ eV}^2, \quad \sin^2 2\theta_\odot = 0.79 \text{ } (\tan^2 \theta_\odot = 0.37), \\ \sin \theta_{13} &= 2.5 \times 10^{-2}, \end{aligned} \quad (51)$$

where $\cos 2\theta_{atm} > 0$ (corresponding to $\Delta > 0$) is chosen, and

$$m_1 = 3.4 \times 10^{-3} \text{ eV}, \quad m_2 = 9.2 \times 10^{-3} \text{ eV}, \quad m_3 = 4.8 \times 10^{-2} \text{ eV}. \quad (52)$$

The mixing angles are determined by $\sin^2 2\theta_{atm} = 1 - 4\Delta^2$ and $\sin^2 2\theta_\odot = 8/(8 + x^2)$ with

$$\Delta = -\frac{\delta m_\nu^{\mu\mu}}{4m_\nu^H}, \quad x = \sqrt{2} \frac{\lambda_2}{A}, \quad (53)$$

as in Eq(5). The requirement of $\sin^2 \theta_\odot \sim 0.8$ (corresponding to $x \sim \sqrt{2}$) results in $\lambda_2 \sim A$.

To see the dependence of the charged-lepton mixing angles, α and β , in neutrino masses and mixings, we perform numerical estimation by keeping the terms proportional to m_μ in Eq.(43) and show the results in Fig.4 - Fig.7. It can be stated that

1. As in Fig.4, since the atmospheric neutrino mixing is dominated by the heavy-lepton contributions, the predictions of $\sin^2 2\theta_{atm}$ and Δm_{atm}^2 are less-dependent on the charged lepton contributions, where their α -dependence hardly manifests itself in this graph because it reflects $1 \leq (c_\alpha + s_\alpha) \leq \sqrt{2}$ in $\delta m_\nu^{e\mu}$ as in Eq.(49), and similarly m_3 is also less-dependent on the charged lepton contributions as in Fig.7;
2. As in Fig.5, since the solar neutrino mixing heavily depends on the charged lepton contributions, the predictions of $\sin^2 2\theta_\odot$ and Δm_\odot^2 permitting the observed values vary with the magnitudes of λ_ℓ and charged-lepton mixing angles;
3. As in Fig.6, since the $\sin \theta_{13}=0$ from the heavy lepton contributions, its magnitude remain suppressed to be $\sin \theta_{13} \lesssim 0.03$ induced by the charged lepton contributions.

From Fig.5, we find that the effects of $m_\mu \neq 0$ in Eq.(51) lower (raise) the magnitude of $\sin^2 2\theta_\odot$ (Δm_\odot^2) from 0.79 ($7.3 \times 10^{-5} \text{ eV}^2$) to 0.76 ($8.3 \times 10^{-5} \text{ eV}^2$). We note that, in the simple-minded case of $\lambda_\ell/\lambda = 1$ instead of $\lambda_\ell/\lambda = 2$ as in Eq.(50), the neutrino oscillations are characterized by

$$\begin{aligned} \Delta m_{atm}^2 &= 2.3 \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{atm} = 0.97, \\ \Delta m_\odot^2 &= 7.0 \times 10^{-5} \text{ eV}^2, \quad \sin^2 2\theta_\odot = 0.85 \quad (\tan^2 \theta_\odot = 0.44), \\ \sin \theta_{13} &= (1.3 - 1.4) \times 10^{-2}, \end{aligned} \tag{54}$$

for the mixing angles of the charged leptons at $\sin 2\beta = 0.75$ with all values of $\sin \alpha (\geq 0)$ allowed.

V. SUMMARY

We have discussed how to theoretically understand the observed properties of neutrino oscillations, which can be summarized as:

1. The atmospheric neutrino mixing is almost maximal and is characterized by $\sin^2 2\theta_{atm} \sim 1$;
2. The solar neutrino mixing deviates from the maximal mixing and points to $\sin^2 2\theta_\odot \sim 0.8$;
3. The U_{e3} -mixing is suppressed;
4. The squared mass differences show the hierarchy of $\Delta m_{atm}^2 \gg \Delta m_\odot^2$.

Our answers consist of

1. The atmospheric neutrino mixing is almost maximal because the dominant contributions to the neutrino mass matrix give

$$M_\nu = \begin{pmatrix} 0 & b & -b \\ b & d & d \\ -b & d & d \end{pmatrix}, \tag{55}$$

as in Eq.(13), which also give $\sin \theta_{13} = 0$, for $c_{23} > 0$ and $s_{23} > 0$ defined in Eq.(3).

2. The solar neutrino mixing deviates from the maximal mixing because less-dominant contributions perturb Eq.(55), yielding $\sin^2 2\theta_\odot \sim 0.8$;
3. The U_{e3} -mixing is suppressed because the dominant contributions ensure the appearance of $U_{e3}=0$ and less-dominant contributions yield the suppressed U_{e3} ;
4. The squared mass differences show the hierarchy of $\Delta m_{atm}^2 \gg \Delta m_\odot^2$ because the tiny breaking of the electron number conservation allows the appearance of the suppressed Δm_\odot^2 .

It should be noted that the following neutrino mass texture instead of Eq.(55):

$$M'_\nu = \begin{pmatrix} 0 & b & b \\ b & d & d \\ b & d & d \end{pmatrix}, \tag{56}$$

yields $\sin \theta_{13} (\sim b/\sqrt{2}d) \neq 0$. These observations are quite different from those based on the two-zero texture of neutrino masses [31].

To realize the above scenario, we have adopted the model based on $SU(3)_L \times U(1)_N$ and have demonstrated that

1. The dominant contributions arise from the two-loop radiative effects of the heavy leptons;
2. The less-dominant contributions arise from the two-loop radiative effects of the ordinary charged leptons.

The model contains the specific Higgs scalars denoted by $\xi=(\xi^{++}, \xi^+, \bar{\xi}^+)^T$ with $L_e=0$ and by $\xi'=(\xi'^{++}, \xi'^+, \bar{\xi}'^+)^T$ with $L_e = -1$, which initiate the two-loop radiative mechanism. These Higgs scalars can also initiate the one-loop radiative mechanism, which turns out to be forbidden by the discrete Z_8 symmetry that simultaneously forbids the appearance of dangerous flavor-changing interactions due to the direct Higgs exchanges inherent to this model. The dominant texture of Eq.(55) is subject to the S_2 -symmetric Yukawa interactions of the heavy leptons involving their diagonalized mass terms. The less-dominant charged lepton contributions receive the S_2 -breaking ones due to the existence of the S_2 -breaking charged-lepton masses. It is argued that, in Eq.(55), the ξ -contributions yield the d -term while the ξ' -contributions yield the b -term, which identically vanishes in the exact L_e -conservation.

The S_2 -permutation symmetry for μ - τ can describe the gross feature of the observed atmospheric neutrino oscillations. Since the existing charged leptons obviously do not respect the S_2 -permutation symmetry, there should be other ingredients that respect the S_2 -permutation symmetry for the neutrino mixings. If we rely upon the seesaw mechanism instead of the radiative mechanism discussed in this article, it is expected that the type II seesaw mechanism [10] is arranged to respect the S_2 -permutation symmetry while the ordinary seesaw mechanism [9] influenced by the charged leptons breaks the S_2 -conservation, which as in the present discussions provides the significant deviation of the solar neutrino mixing from the maximal one.

APPENDIX A: CHARGED LEPTON MASSES

In this appendix, the diagonalization of the lepton mass matrix M_ℓ is demonstrated. The diagonal masses are obtained after the transformation by $M_\ell^{diag} = \text{diag.}(m_e, m_\mu, m_\tau) = U_\ell^\dagger M_\ell V_\ell$, where unitary matrices U_ℓ and V_ℓ are given by

$$U_\ell = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & s_\alpha \\ 0 & -s_\alpha & c_\alpha \end{pmatrix}, \quad V_\ell = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\beta & s_\beta \\ 0 & -s_\beta & c_\beta \end{pmatrix} \quad (\text{A1})$$

with $c_\alpha = \cos \alpha$, etc., defined by

$$\begin{aligned} c_\alpha &= \sqrt{\frac{(m_\ell^{\tau\tau})^2 + (m_\ell^{\tau\mu})^2 - m_\mu^2}{m_\tau^2 - m_\mu^2}}, & s_\alpha &= \sqrt{\frac{-(m_\ell^{\tau\tau})^2 - (m_\ell^{\tau\mu})^2 + m_\tau^2}{m_\tau^2 - m_\mu^2}}, \\ c_\beta &= \sqrt{\frac{(m_\ell^{\tau\tau})^2 + (m_\ell^{\mu\tau})^2 - m_\mu^2}{m_\tau^2 - m_\mu^2}}, & s_\beta &= \sqrt{\frac{-(m_\ell^{\tau\tau})^2 - (m_\ell^{\mu\tau})^2 + m_\tau^2}{m_\tau^2 - m_\mu^2}}, \end{aligned} \quad (\text{A2})$$

where $m_\mu^2 + m_\tau^2 = (m_\ell^{\mu\mu})^2 + (m_\ell^{\tau\tau})^2 + (m_\ell^{\mu\tau})^2 + (m_\ell^{\tau\mu})^2$ is a obvious relation. The diagonal masses are computed to be:

$$\begin{aligned} m_e^2 &= (m_\ell^{ee})^2, \\ m_\mu^2 &= \frac{1}{2} [(m_\ell^{\tau\tau})^2 + (m_\ell^{\mu\mu})^2 + (m_\ell^{\tau\mu})^2 + (m_\ell^{\mu\tau})^2 - M^2], \\ m_\tau^2 &= \frac{1}{2} [(m_\ell^{\tau\tau})^2 + (m_\ell^{\mu\mu})^2 + (m_\ell^{\tau\mu})^2 + (m_\ell^{\mu\tau})^2 + M^2], \end{aligned} \quad (\text{A3})$$

where

$$\begin{aligned} M^4 &= [(m_\ell^{\tau\tau})^2 - (m_\ell^{\mu\mu})^2]^2 + [(m_\ell^{\tau\mu})^2 - (m_\ell^{\mu\tau})^2]^2 \\ &\quad + 2(m_\ell^{\tau\tau} m_\ell^{\mu\tau} + m_\ell^{\mu\mu} m_\ell^{\tau\mu})^2 + 2(m_\ell^{\tau\tau} m_\ell^{\tau\mu} + m_\ell^{\mu\mu} m_\ell^{\mu\tau})^2. \end{aligned} \quad (\text{A4})$$

There are the following relations for non-diagonal and diagonal charged lepton masses:

$$\begin{aligned} m_\ell^{\mu\mu} &= S^2 m_\tau + C^2 m_\mu, & m_\ell^{\tau\tau} &= C^2 m_\tau + S^2 m_\mu, \\ m_\ell^{\mu\tau} &= \frac{1}{c_\beta^2 - s_\alpha^2} [(c_\alpha s_\alpha C^2 - c_\beta s_\beta S^2) m_\tau - (c_\beta s_\beta C^2 - c_\alpha s_\alpha S^2) m_\mu], \\ m_\ell^{\tau\mu} &= \frac{1}{c_\beta^2 - s_\alpha^2} [(c_\beta s_\beta C^2 - c_\alpha s_\alpha S^2) m_\tau - (c_\alpha s_\alpha C^2 - c_\beta s_\beta S^2) m_\mu], \end{aligned} \quad (\text{A5})$$

where C^2 and S^2 are defined by

$$C^2 = \frac{c_\alpha^2 + c_\beta^2}{2}, \quad S^2 = \frac{s_\alpha^2 + s_\beta^2}{2}. \quad (\text{A6})$$

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Table Captions

TABLE I: Particle contents with S_{L2} , Z_8 and L_e quantum numbers for quarks, leptons and Higgs scalars, where $S_{2L} = +(-)$ denotes symmetric (antisymmetric) states and $\omega^8=1$.

Figure Captions

FIG.1: Forbidden one-lopp diagram.

FIG.2: Heavy-lepton mediated two-loop diagrams.

FIG.3: Charged-lepton mediated two-loop diagrams.

FIG.4: The dependences of $\sin^2 2\theta_{atm}$ (the black curves) and Δm_{atm}^2 (the gray curves) in functions of $\sin 2\beta$ for the given values of $\lambda_\ell/\lambda=1.0, 1.5, 2.0$, where the dependence in $\sin \alpha$ is so weak that it is covered by the line thickness.

FIG.5: The dependences of $\sin^2 2\theta_\odot$ (the black curves) and Δm_\odot^2 (the gray curves) in functions of $\sin 2\beta$ for the given values of $\sin \alpha=0$ (thick curves), 1 (thin curves) and $\lambda_\ell/\lambda=1.0, 1.5, 2.0$.

FIG.6: The dependences of $\sin \theta_{13}$ as a function of $\sin 2\beta$ for the given values of $\sin \alpha=0$ (thick curves), 1 (thin curves) and $\lambda_\ell/\lambda=1.0, 1.5, 2.0$.

FIG.7: The dependences of $m_{1,2,3}$ as a function of $\sin 2\beta$ for the given values of $\lambda_\ell/\lambda=1.0, 1.5, 2.0$, where the dependence in $\sin \alpha$ is so weak that it is covered by the line thickness.

TABLE I: Particle contents with S_{L2} , Z_8 and L_e quantum numbers for quarks, leptons and Higgs scalars, where $S_{2L} = +(-)$ denotes symmetric (antisymmetric) states and $\omega^8=1$.

	ψ_L^e	e_R	ψ_{+L}	ℓ_{+R}	ψ_{-L}	ℓ_{-R}	E_R^e	E_{+R}	E_{-R}
S_{2L}	+	+	+	+	-	-	+	+	-
Z_8	ω^6	ω^4	ω^5	ω^3	1	ω^6	ω^3	ω^2	ω^7
L	1	1	1	1	1	1	1	1	1
L_e	1	1	0	0	0	0	1	0	0

	η	ρ	ρ'	ρ''	χ	χ'	ξ	ξ'	k^{++}	k'^{++}
S_{2L}	+	+	-	-	+	+	-	+	+	+
Z_8	ω	ω^2	ω^7	ω^5	ω^3	ω	ω^3	ω^5	ω^2	ω^4
L	0	0	0	0	0	0	-2	-2	-2	-2
L_e	0	0	0	0	0	0	0	-1	0	0

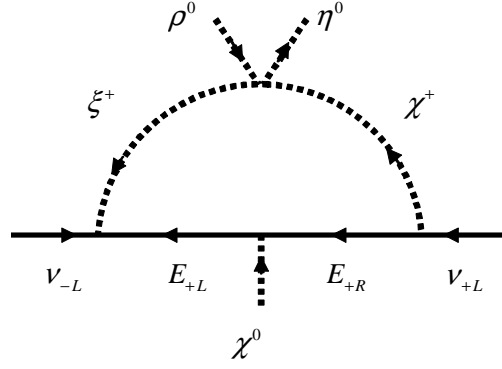


FIG. 1: Forbidden one-loop diagram.

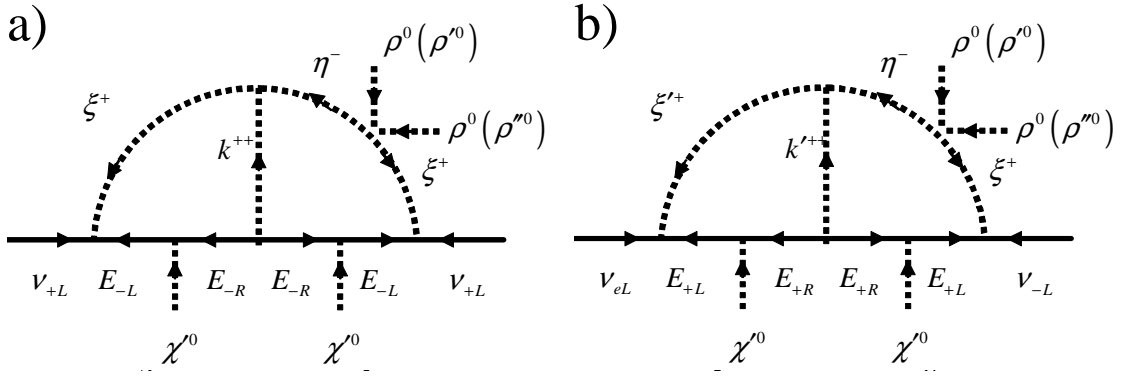


FIG. 2: Heavy-lepton mediated two-loop diagrams.

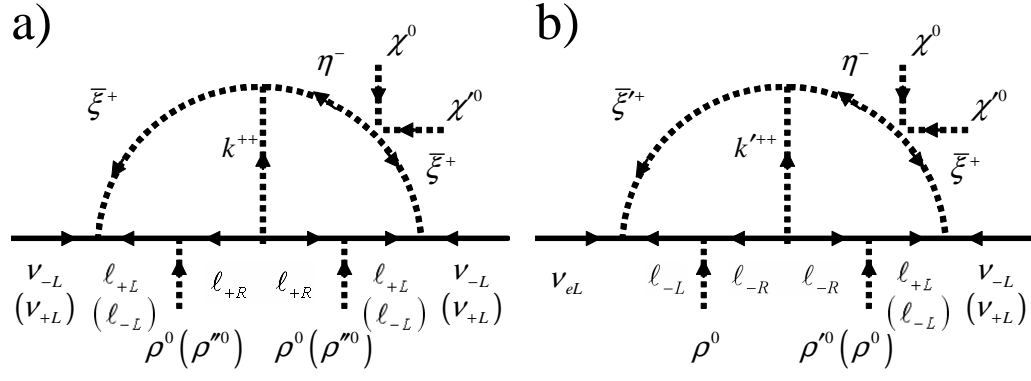


FIG. 3: Charged-lepton mediated two-loop diagrams.

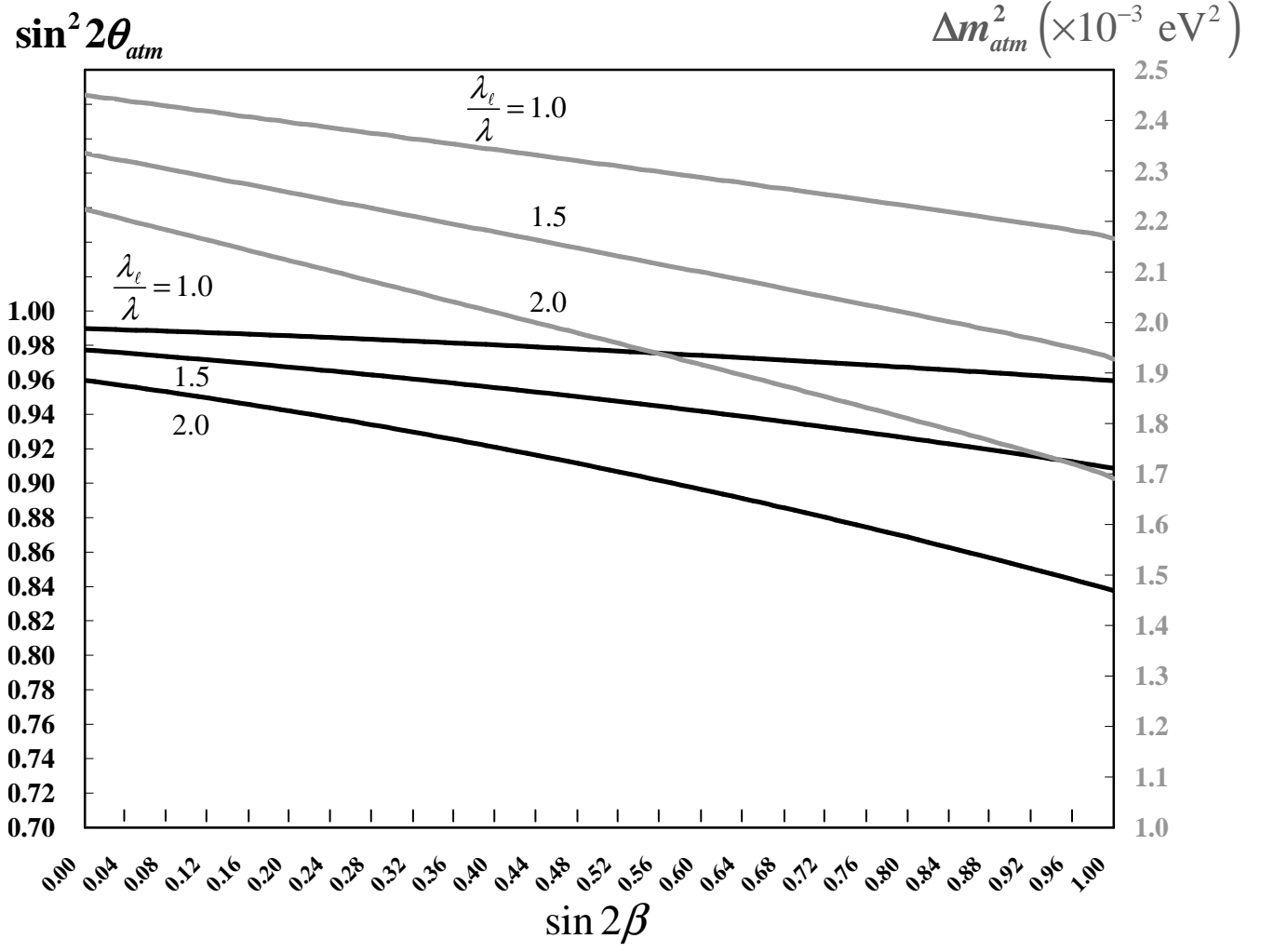


FIG. 4: The dependences of $\sin^2 2\theta_{atm}$ (the black curves) and Δm_{atm}^2 (the gray curves) in functions of $\sin 2\beta$ for the given values of $\lambda_\ell/\lambda=1.0, 1.5, 2.0$, where the dependence in $\sin \alpha$ is so weak that it is covered by the line thickness.

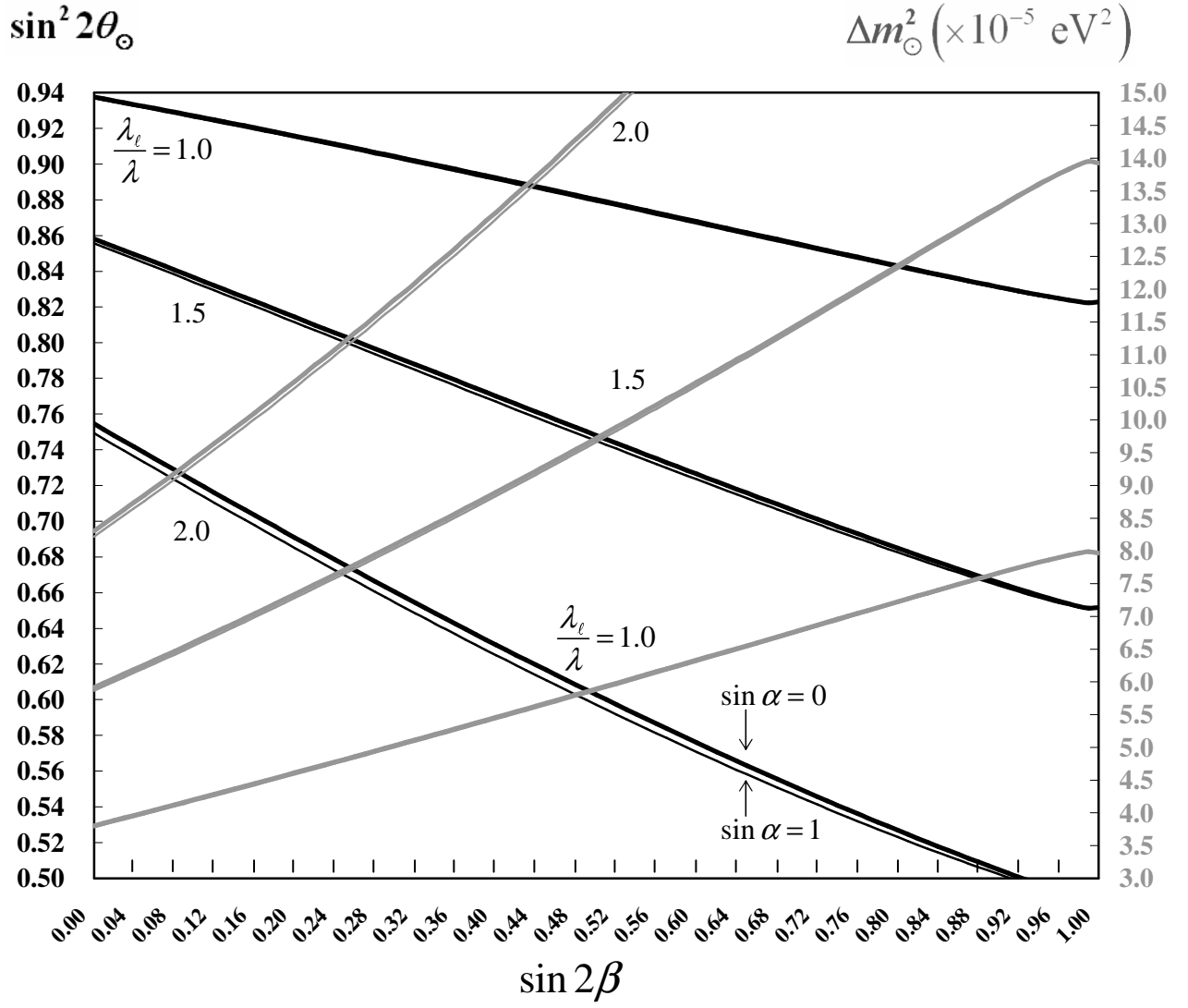


FIG. 5: The dependences of $\sin^2 2\theta_\odot$ (the black curves) and Δm_\odot^2 (the gray curves) in functions of $\sin 2\beta$ for the given values of $\sin \alpha=0$ (thick curves), 1 (thin curves) and $\lambda_\ell/\lambda=1.0, 1.5, 2.0$.

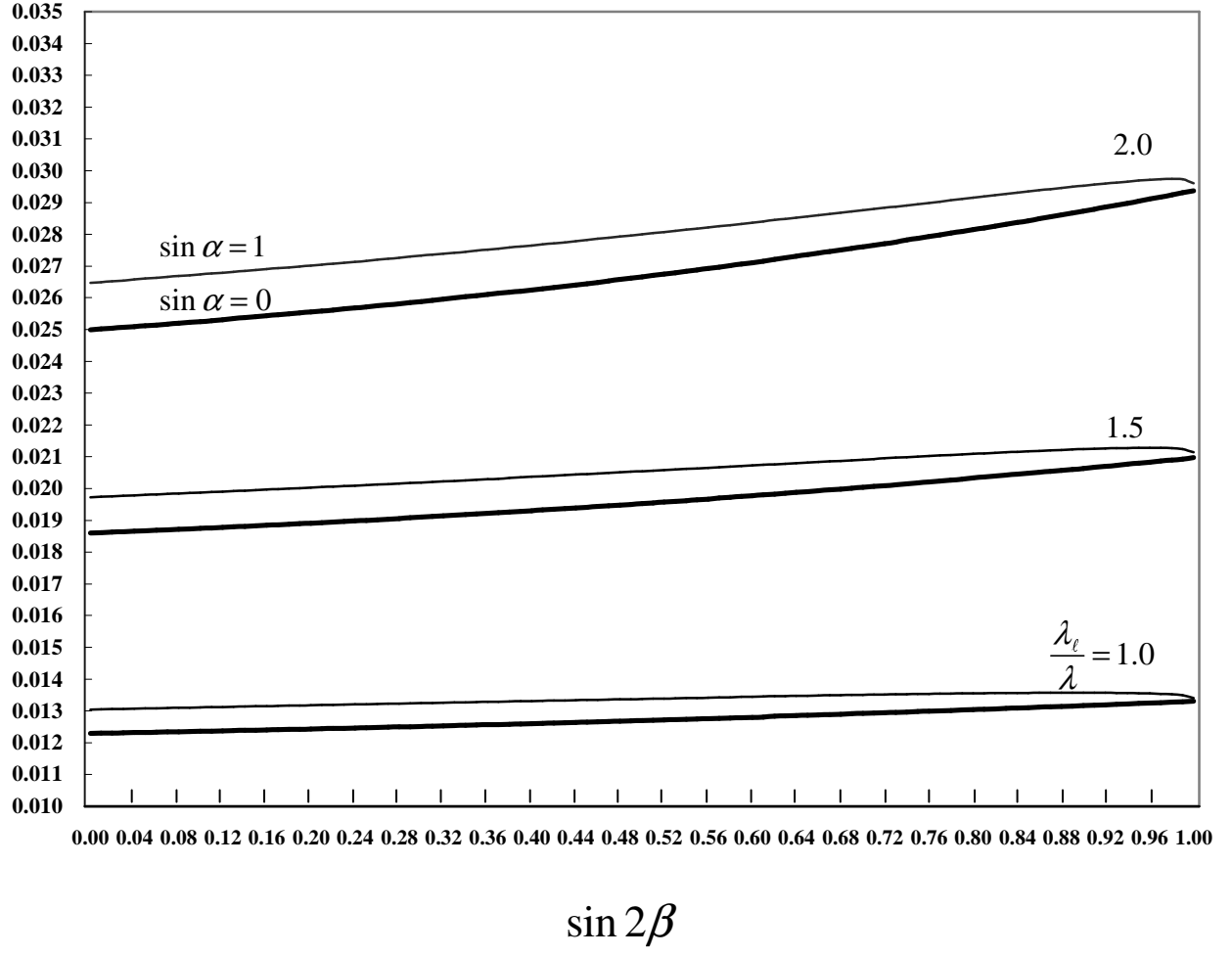
$\sin \theta_{13}$ 

FIG. 6: The dependences of $\sin \theta_{13}$ as a function of $\sin 2\beta$ for the given values of $\sin \alpha=0$ (thick curves), 1 (thin curves) and $\lambda_\ell/\lambda=1.0, 1.5, 2.0$.

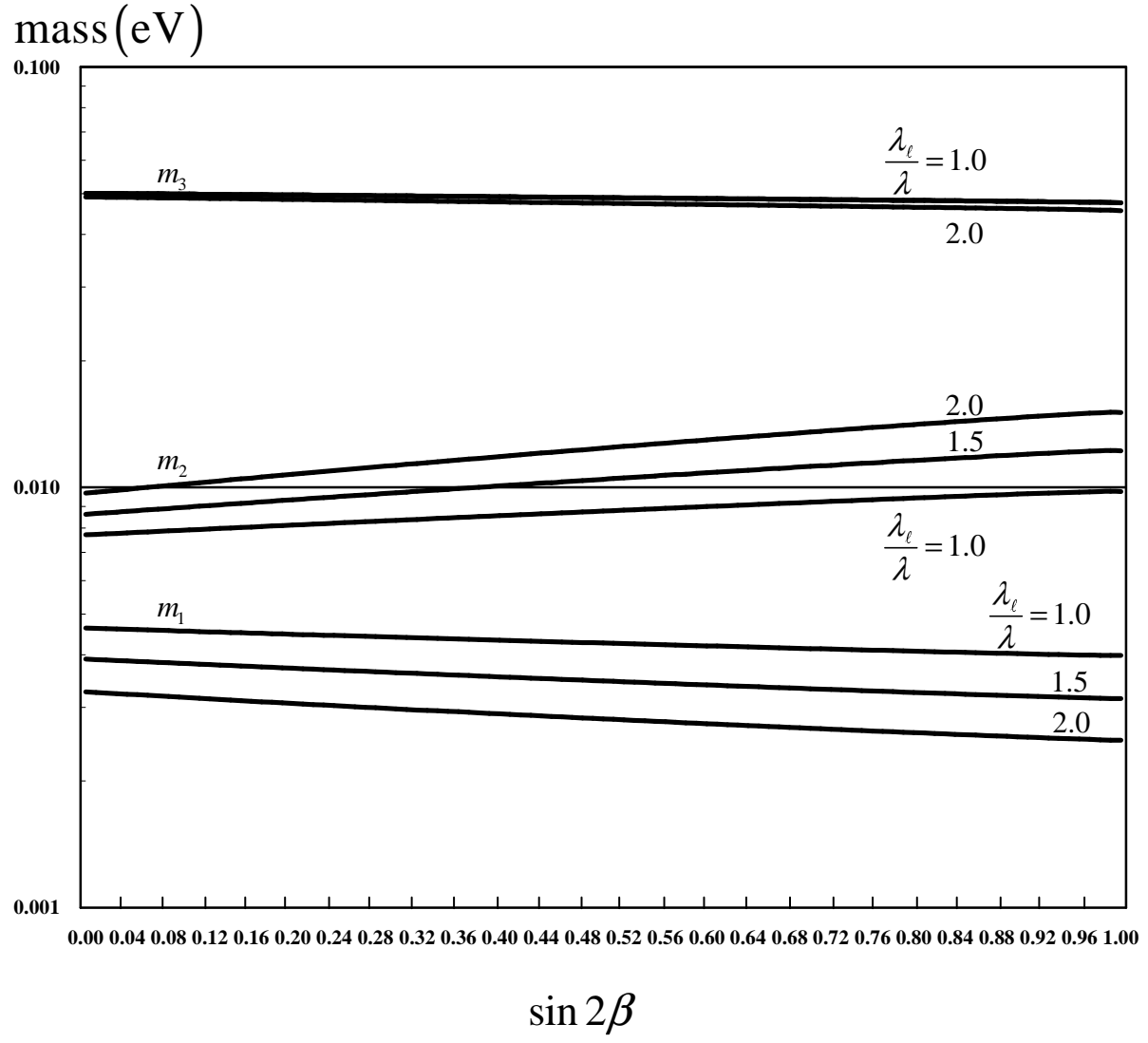


FIG. 7: The dependences of $m_{1,2,3}$ as a function of $\sin 2\beta$ for the given values of $\lambda_\ell/\lambda=1.0, 1.5, 2.0$, where the dependence in $\sin \alpha$ is so weak that it is covered by the line thickness.